Soliton stability in a generalized sine-Gordon potential

Rubén Cordero* and Roberto D. Mota†

Abstract

We study stability of a generalized sine-Gordon model with two coupled scalar fields in two dimensions. Topological soliton solutions are found from the first-order equations that solve the equations of motion. The perturbation equations can be cast in terms of a Schrödinger-like operators for fluctuations and their spectra are calculated.

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1 Introduction

It is well known that in field theories when a discrete symmetry is broken domain walls arise. Domain walls have been observed in condensed matter, for example, in liquid crystals. In the cosmological context domain walls could appear in phase transitions in the early universe and have some important consequences (Vilenkin and Shellard, 1994).

In the domain walls context, there exist classical static configurations with finite minimum localized energy, see for instance (Bogomol'nyi, 1976; Prasad, 1975). Several authors have been interested in coupled scalar fields systems due to their important physical properties. For example, Peter showed in (Peter, 1996) that surface current-carrying domain wall arises when a bosonic charge carrier is coupled to the Higgs field forming the wall. In (Bazeia, et al., 1997; Riazi, et al., 2001) was studied linear stability of soliton solutions for

*Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, Ed. 9, Unidad Profesional Adolfo López Mateos, 07738 México D F, Mexico. cordero@esfm.ipn.mx.

†Unidad Profesional Interdisciplinaria de Ingeniería y Tecnologías Avanzadas, IPN. Av. Instituto Politécnico Nacional 2580, Col. La Laguna Ticomán, Delegación Gustavo A. Madero, 07340 México D. F. Mexico. mota@esfm.ipn.mx

a class of systems of coupled scalar self-interacting fields following the standard approach of classical stability.

Besides, as it was extensively shown, Witten's supersymmetric quantum mechanics (Witten, 1981) is the most recent way to study solvable as well as perturbative problems (Witten, 1981; Lahiri et al., 1990; Cooper et al., 1995; Junker, 1996; Cooper et al., 2001).

Some systems of coupled scalar fields present soliton solutions and their linear stability have been addressed by means of the SUSY QM formalism in a 2×2 superpotential realization (Bazeia et al., 1995; Bazeia and Santos, 1996; de Lima Rodrigues et al., 1998; Dias at al., 2002). The superpotential satisfies the Riccati equation associated to the perturbation Hessian and establishes the stability of the system. This fact is because of the corresponding supersymmetric operators factorize the perturbation equation and automatically ensure non-negative equal perturbation frequencies (Bazeia and Santos, 1996).

The sine-Gordon system has been applied to a wide class of physical problems like propagation of crystal dislocations, two-dimensional models of elementary particles, propagation of splay waves in membranes, Bloch wall motion in magnetic crystals and magnetic flux in Josephson lines (Rajaraman, 1982). It is well known that this system in (1+1) dimensions has classical soliton solutions and their non-dissipative properties could be explained like a finely-tuned balance between self-interactions and dispersion.

A generalization of sine-Gordon system with two coupled real scalar fields showed an important and rich behavior (Riazi et al., 2002). The potential of this system consists of a product between a trigonometric and polynomial functions of the fields. Depending on the rest energies and the boundary conditions, the spectrum of solitons could be stable, unstable or meta-stable. The former classical stability analysis was established by means of numerical analysis.

Another generalization of coupled sine-Gordon model has been given as an example of continuously degenerate soliton (Shifman and Voloshin, 1998). In contrast to the models mention above, this model involves a highly coupled self-interacting fields with non-polynomial form. This model has richer structure and dynamics and deserves further analysis.

In this paper we advocate to the study of linear stability approximation of the generalized sine-Gordon model proposed in (Shifman and Voloshin, 1998). In section 2, we give the model consisting of two couple real scalar fields, we present the first order differential

equations that minimize the energy for the fields and find a particular solution of them. In section 3, we show that the stability equations can be analyzed in terms of SUSY QM formalism and reduce the problem of stability to solve a Riccati equation associated to the perturbation Hessian. In section 4, we find the spectra of the fluctuation operator and explictly show the stability of the soliton solution. Finally, we give the conclusions of this work.

2 The Model

In this paper we consider the generalization of the sine-Gordon model for two scalar interacting fields given by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)^2 + \frac{1}{2}(\partial_\mu X)^2 - \frac{1}{2}\cos^2\Phi(1+\alpha\sin X)^2 - \frac{1}{2}\cos^2 X(1+\alpha\sin\Phi)^2, \tag{1}$$

where α is a dimensional parameter, and all other dimensional parameters are set equal to unity. For $0 \leq \Phi, X \leq 2\pi$, the potential in (1) has three minima at $\Phi = X = \pi/2$; $\Phi = \pi/2$, $X = 3\pi/2$ and $\Phi = X = 3\pi/2$, one maximum at $\Phi = 0$, $X = \pi/2$, and three saddle points at $\Phi = \pi/2$, $X = \pi$; $\Phi = \pi/2$, $X = 0$ and $\Phi = 3\pi/2$, $X = \pi/2$.

The equations of motion for the model (1) are the usual ones:

$$\Box\Phi + \frac{\partial}{\partial\Phi}V = 0, \qquad\qquad \Box X + \frac{\partial}{\partial X}V = 0 \tag{2}$$

which become for a static configurations

$$\begin{aligned} \Phi'' &= -\cos\Phi\sin\Phi(1+\alpha\sin X)^2 + \alpha\cos^2 X(1+\alpha\sin X)\cos\Phi & (3)\\ X'' &= -\cos X\sin X(1+\alpha\sin\Phi)^2 + \alpha\cos^2\Phi(1+\alpha\sin\Phi)\cos X, & (4)\end{aligned}$$

where primes means derivatives with respect to space variable.

The form of the energy of the system can be written as

$$E_s = \int_{-\infty}^{\infty}\left[\left(\frac{d\Phi}{dz} - W_\Phi\right)^2 + \left(\frac{dX}{dz} - W_X\right)^2\right]dz + \left|\int_{-\infty}^{\infty}\frac{\partial}{\partial z}W[\Phi(z),X(z)]dz\right| \tag{5}$$

where $W[\Phi(z), X(z)]$ is the corresponding superpotential of (1), which turns out to be

$$W = -\sin\Phi - \sin X - \alpha(\sin\Phi)(\sin X). \tag{6}$$

In (Shifman and Voloshin, 1998) this superpotential was referred as a generalization of the sine-Gordon model. It is periodic in both Φ and X; for $\alpha = 0$ it describes two decoupled

fields, representing each of them a supergeneralization of the sine-Gordon model. If $\alpha \neq 0$ the fields Φ and X start interacting with each other. Inside the periodicity domain $0 \leq \Phi, X \leq 2\pi$, $-W$ has one maximum at $\Phi = X = \pi/2$, one minimum at $\Phi = X = 3\pi/2$ and two saddle points at $\Phi = \pi/2$, $X = 3\pi/2$ and $\Phi = 3\pi/2$, $\pi/2$ at least for small values of α.

The lower bound for the energy is achieved if Φ and X satisfy

$$\begin{aligned} \Phi' &= -\cos\Phi(1+\alpha\sin X) \\ X' &= -\cos X(1+\alpha\sin\Phi). \end{aligned} \tag{7}$$

For the case $X = \pi/2$, we have

$$\frac{d\Phi}{dz} = -\cos\Phi(1+\alpha), \tag{8}$$

whose solution is

$$\Phi = -\tan^{-1}\left(\frac{c^2 e^{z(1+\alpha)} - e^{-z(1+\alpha)}}{2}\right). \tag{9}$$

Other possible solution of equations (7) is obtained for $X = 3\pi/2$ and the solution is obtained from the former equation by substituting α by $-\alpha$. Interchanging the fields X and Φ in the last equations we get the solution for $\Phi = \pi/2$ or $\Phi = 3\pi/2$.

We attempted to find the general solutions of the coupled equations (7) by the trial orbit method of Rajaraman (Rajaraman, 1977), however, because of the difficulty of the system we were unable to find them.

3 Stability Equations and SUSY QM

We are interested in determining the classical stability of this system under small fluctuations around a static configuration. In order to investigate the linear stability of the interacting fields we proceed in the usual way by considering small perturbations around the static scalar fields

$$\Phi(z,t) = \Phi(z) + \eta(z,t) \tag{10}$$

$$X(z,t) = X(z) + \xi(z,t). \tag{11}$$

The stability equations can be written in a Schrödinger-like equation

$$S_l \Psi_n = {\omega_n}^2 \Psi_n \tag{12}$$

where $n = 0, 1, 2...$ The differential operator S_l is given by

$$S_l = \begin{pmatrix} -\frac{d^2}{dz^2} + \frac{\partial^2}{\partial \Phi^2} V & \frac{\partial^2}{\partial \Phi \partial X} V \\ \frac{\partial^2}{\partial \Phi \partial X} V & -\frac{d^2}{dz^2} + \frac{\partial^2}{\partial X^2} V \end{pmatrix}_{|\Phi=\Phi(z), X=X(z)} \equiv -\frac{d^2}{dz^2}\mathbf{I}_{2\times 2} + \mathbf{V}_{PH} \tag{13}$$

and the two components wave functions are

$$\Psi_n = \begin{pmatrix} \Phi_n(z) \\ X_n(z) \end{pmatrix}, \tag{14}$$

where we have expanded the fluctuations $\alpha(z,t)$ and $\beta(z,t)$ in terms of normal modes

$$\eta(z,t) = \sum_n a_n \eta_n(z) e^{i\omega_n t} \tag{15}$$

$$\xi(z,t) = \sum_n b_n \xi_n(z) e^{i\omega_n t}. \tag{16}$$

Notice that in the case when the differential operator S_l is diagonal the perturbation fields could be expanded in terms of different frequencies.

The SUSY QM approach to linear stability consists in realizing a 2×2-matrix superpotential, which is obtained by solving the Riccati equation associated to the perturbation Hessian $\mathbf{V}_{PH}$

$$\mathbf{W}^2 + \mathbf{W}' = \mathbf{V}_{PH}. \tag{17}$$

The existence of $\mathbf{W}$ that satisfies this equation ensures the existence of the first order self-adjoint differential operators

$$\mathcal{D}^{\pm} = \pm \mathbf{I}\frac{d}{dz} + \mathbf{W}(z) \tag{18}$$

that factorize the operator $S_l = \mathcal{D}^+\mathcal{D}^-$. This fact implies the stability for equal fluctuation frequencies, since $0 \leq |\mathcal{D}^-\Psi_n|^2 = (\mathcal{D}^-\Psi_n)^\dagger(\mathcal{D}^-\Psi_n) = \langle \mathcal{D}^+\mathcal{D}^- \rangle = \langle S_l \rangle = \omega_n^2$.

For our case the matrix elements of $\mathbf{V}_{PH}$ are given by

$$\begin{aligned} (\mathbf{V}_{PH})_{11} &= -(\cos^2\Phi - \sin^2\Phi)(1+\alpha\sin X)^2 + \alpha^2\cos^2 X\cos^2\Phi \\ &- \alpha\cos^2 X(1+\alpha\sin\Phi)\sin\Phi \\ (\mathbf{V}_{PH})_{22} &= -(\cos^2 X - \sin^2 X)(1+\alpha\sin\Phi)^2 + \alpha^2\cos^2\Phi\cos^2 X \\ &- \alpha\cos^2\Phi(1+\alpha\sin X)\sin X \end{aligned} \tag{19}$$

$$\begin{aligned} (\mathbf{V}_{PH})_{12} &= (\mathbf{V}_{PH})_{21} = -2\alpha\cos\Phi\sin\Phi(1+\alpha\sin X)\cos X \\ &- 2\alpha\cos X\sin X(1+\alpha\sin\Phi)\cos\Phi, \end{aligned} \tag{20}$$

The solution of Riccati equation (17) for configurations satisfying equations (7) is

$$\mathbf{W}_{min} = \begin{pmatrix} (1+\alpha\sin X)\sin\Phi & -\alpha\cos\Phi\cos X \\ -\alpha\cos\Phi\cos X & (1+\alpha\sin\Phi)\sin X \end{pmatrix}. \tag{21}$$

For the sector $X = \pi/2$, the fluctuation potential term becomes

$$\mathbf{V}_{min} = \begin{pmatrix} -(1+\alpha)^2(\cos^2\Phi - \sin^2\Phi) & 0 \\ 0 & (1+\alpha\sin\Phi)^2 - \alpha\cos^2\Phi(1+\alpha) \end{pmatrix}, \tag{22}$$

so, the corresponding superpotential is

$$\mathbf{W}_{min} = \begin{pmatrix} (1+\alpha)\sin\Phi & 0 \\ 0 & (1+\alpha\sin\Phi) \end{pmatrix}. \tag{23}$$

We point out the existence of another self-adjoint and non-negative second-order differential operator $S_l' = \mathcal{D}^-\mathcal{D}^+$ which plays the role of the supersymmetric partner operator of S_l in SUSY QM. The operators S_l and S_l' have the same energy spectrum except for the ground state.

4 Spectrum of the Second Order Fluctuation Operator

The study of stability for the general case is very difficult. However, in order to have analytical results in the case of $X = \pi/2$ (the results we are going to obtain are automatically true for $\Phi = \pi/2$), we take the particular case of $c = 1$ in equation (9) *i. e.* $\tan\Phi = \sinh z(1+\alpha)$. Since the differential operator S_l is diagonal we could have different fluctuation frequencies that can be determined from the perturbation equations

$$-\frac{d^2\eta_n}{dz^2} - (1+\alpha)\left(2\mathrm{sech}^2 z(1+\alpha) - 1\right)\eta_n = \omega_n^2\eta_n \tag{24}$$

and

$$-\frac{d^2\xi_n}{dz^2} + \left(1+\alpha^2 - 2\alpha\tanh z(1+\alpha) - \alpha(1+2\alpha)\mathrm{sech}^2 z(1+\alpha)\right)\xi_n = \omega_n^2\xi_n. \tag{25}$$

Performing the variable change $y = z(1+\alpha)$, equation (24) transforms to the Rosen-Morse equation (Morse and Feshbach, 1953). We find that the fluctuation frequencies are

$$\omega_n^2 = (1+\alpha)^2\left(1-(1-n)^2\right)^2. \tag{26}$$

However the bound states exist only for $n < 1$ (Morse and Feshbach, 1953). Thus, the ground state $\eta_0 = (1+\alpha)\text{sech}z(1+\alpha)$ with eigenvalue $\omega_0 = 0$ is stable.

By means of the same variable change the equation (25) can be cast as a Rosen-Morse equation whose eigenvalues are

$$\begin{aligned} \omega_n^2 &= 1+\alpha^2-(1+\alpha)^2\left[\left(\frac{3\alpha+1}{2(1+\alpha)}-(n+1/2)\right)^2\right] \\ &- \frac{4\alpha}{(1+\alpha)^2\left(\frac{3\alpha+1}{2(1+\alpha)}-(n+1/2)\right)^2} \end{aligned} \tag{27}$$

which are the frequencies for possible bound states. However, for this case we have no bound states because n must be less than zero for both $\alpha > 0$ and $\alpha < 0$ (Morse and Feshbach, 1953).

This means that the solution configurations are stable under small perturbations around $X = \pi/2$, $\tan\Phi = -\sinh z(1+\alpha)$ (the same is true for $\Phi = \pi/2$, $\tan X = -\sinh z(1+\alpha)$).

Conclusions

We have applied the SUSY QM formalism to study the linear stability of the Shifman generalization of the sine-Gordon model. We have shown that stability for soliton configurations is ensured by solving the Riccati equation for the 2×2 superpotential associated to the non-diagonal perturbation Hessian. The spectrum of the second order fluctuation operator for the general case is very difficult to find it. Thus, we have got the fluctuation spectrum for the particular case $X = \pi/2$ (or $\Phi = \pi/2$), and we have found analytical solutions and explicitly found that the system is stable. We notice that equations (24) and (25) can be reduced to a Rosen-Morse equation. On another hand, the Rosen-Morse equation have been studied from the shape invariance approach of SUSY QM (Dutt et al., 1988). Thus, each one of the equations (24) and (25) has a scalar superpotential. Therefore, we have given a complete treatment of linear stability for the generalized sine-Gordon superpotential from the point of view of SUSY QM.

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